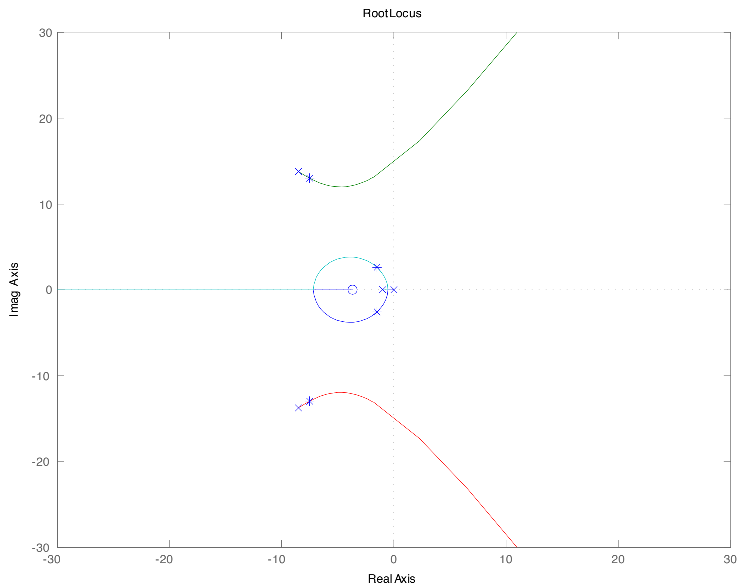
(c) $L = \mathsf{place}(\mathsf{F}', \mathsf{H}', \mathsf{roots}([1\ 2 * \mathsf{zeta} * \mathsf{wn}\ \mathsf{wn\hat{}2}]))' = [14\ 211]^T$.

(d) The transfer function for the controller is,

$$\begin{aligned} D_c(s) &= -\mathbf{K}(s\mathbf{I} - \mathbf{F} + \mathbf{GK} + \mathbf{LH})^{-1}\mathbf{L} \\ &= \frac{-(54.8s + 202.5)}{s^2 + 17s + 262}. \end{aligned}$$

(e) The figure below shows the root locus around a nominal gain of 10, which is indicated by asterisk.



Problem 7.50: Root locus of the closed-loop system as plant gain is varied.

51. Unstable equations of motion of the form,

$$\ddot{x} = x + u,$$

arise in situations where the motion of an upside-down pendulum (such as a rocket) must be controlled.
a) Let $u = -Kx$ (position feedback alone), and sketch the root locus with respect to the scalar gain K.
b) Consider a lead compensator of the form,

$$U(s) = K\frac{s+a}{s+10}X(s).$$

Select a and K so that the system will display a rise time of about 2 sec and no more than 25% overshoot. Sketch the root locus with respect to K.
c) Sketch the Bode plot (both magnitude and phase) of the uncompensated plant.
d) Sketch the Bode plot of the compensated design, and estimate the phase margin.
e) Design state feedback so that the closed-loop poles are at the same locations as those of the

design in part (b).
f) Design an estimator for x and $\dot{x}$ using the measurement of $x = y$, and select the observer gain $\mathbf{L}$ so that the equation for $\tilde{x}$ has characteristic roots with a damping ratio $\zeta = 0.5$ and a natural frequency $\omega_n = 8$ rad/sec.
g) Draw a block diagram of your combined estimator and control law, and indicate where $\hat{x}$ and $\dot{x}$ appear. Draw a Bode plot for the closed-loop system, and compare the resulting bandwidth and stability margins with those obtained using the design of part (b).

Solution:

(a) The root locus using position feedback alone is shown below. Notice that no matter how large the gain is made, the closed-loop roots are never strictly in the LHP.

(b) First of all, we need to translate the specifications into values for ω_n and ζ. Although the closed system with a lead compensator is third-order, we assume the rules of thumb for a second-order system are valid and then validate our design after settling on values for a and K.

$$M_p < 25\% \Longrightarrow \zeta > 0.4, \omega_n = \frac{1.8}{t_r} = \frac{1.8}{2} = 0.9.$$

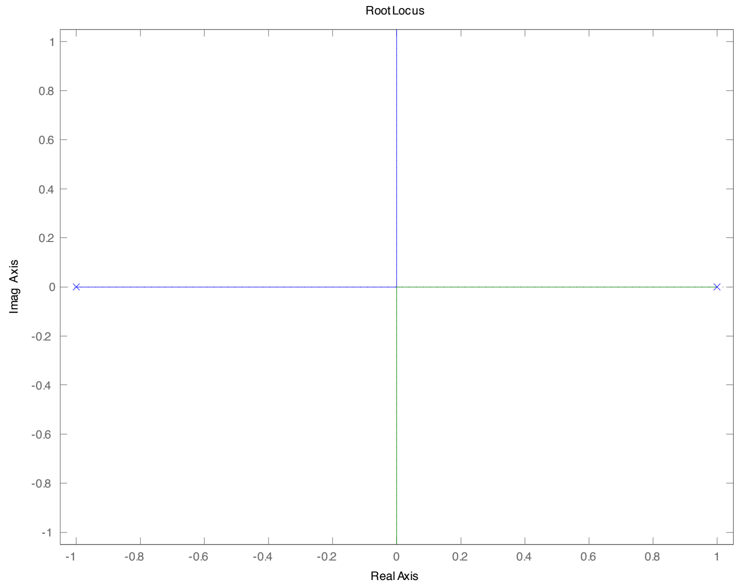
Try $\zeta = 0.4$ and $\omega_n = 1$ for the design. Because the form of the compensator is specified, we can calculate the closed-loop transfer function to be,

$$\frac{Y(s)}{R(s)} = T(s) = \frac{s+10}{s^3 + 10s^2 + (K-1)s + (Ka - 10)}.$$

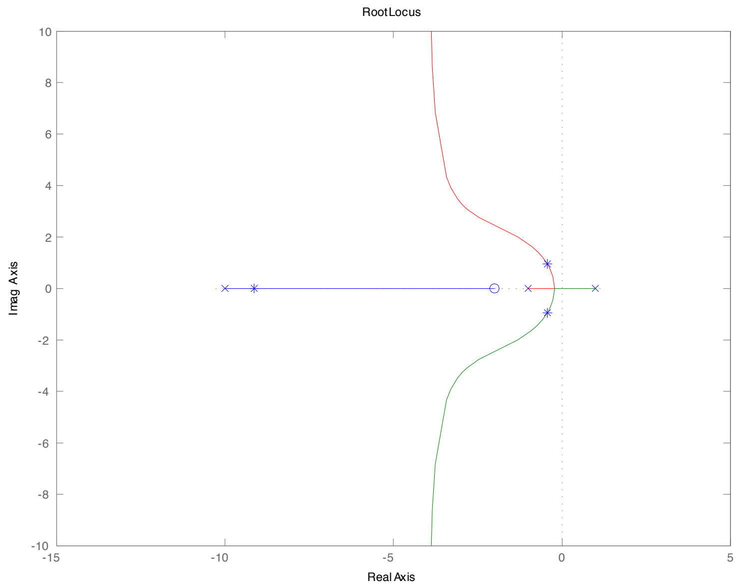
Note that we have subtly introduced r as a reference input to the plant. The desired closed loop poles should be placed at (taking $\alpha = 10$),

$$(s+\alpha)(s^2 + 2\zeta\omega_n s + \omega_n^2) = (s+10)(s^2 + 0.8s + 1) = s^3 + 10.8s^2 + 9s + 10.$$

Although the coefficient for the s^2 term doesn't match exactly, we just want to get a ballpark estimate for K and a. So comparing the other coefficients, we find $K = 10$ and $a = 2$. Using these values, the root locus for using the lead compensator is shown. To verify that our design is acceptable, we also check the step response of the system. This is shown on the last figure in this section.



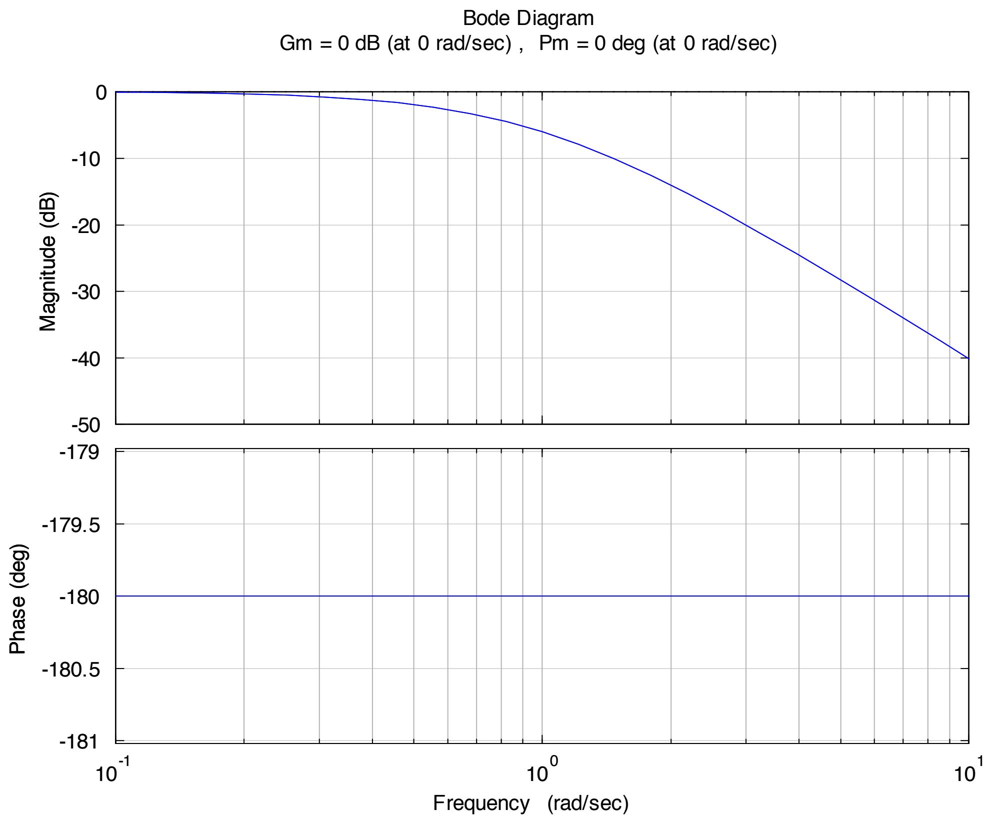
Problem 7.51: Root locus with position feedback alone.



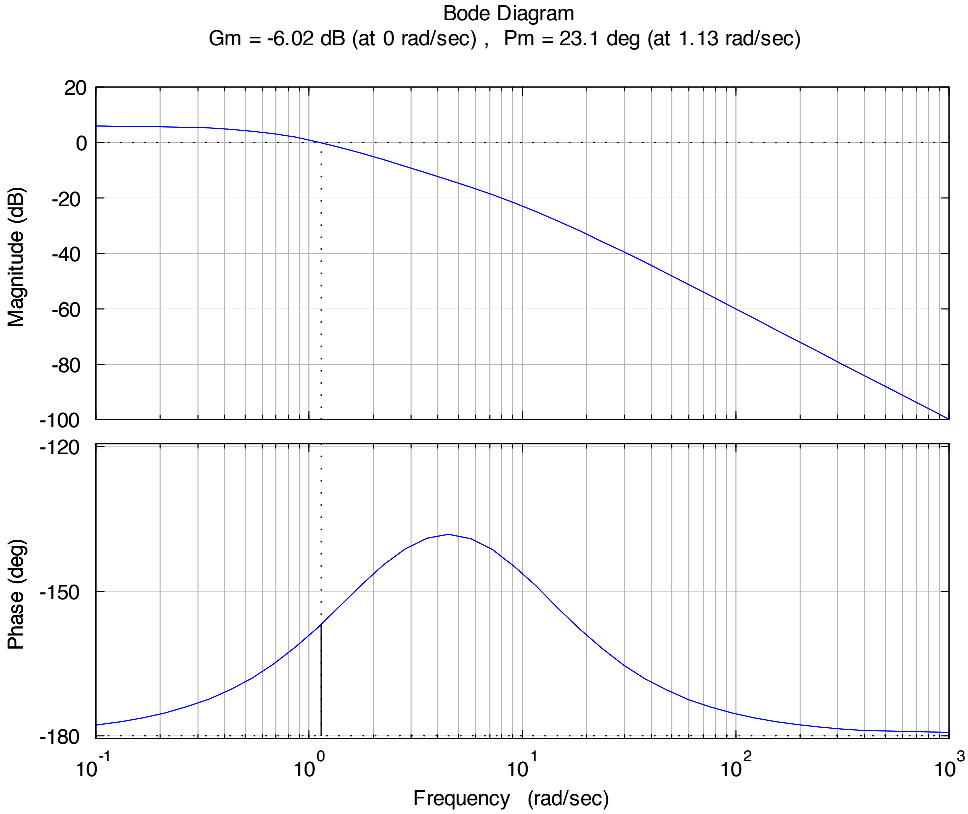
Root locus for Problem 7.51.

(c) The Bode plot of $\frac{1}{s^2-1}$ is shown below.

(d) The Bode plot of the compensated design is also shown on the next page. The phase margin is approximately 23°. The gain margin is 0.5.



Problem 7.51: Bode plots for the open-loop system.



Problem 7.51: Compensator and plant combined.

(e) Although the design in part (b) has three closed-loop poles (due to the lead compensator), full state feedback on a second-order system does not introduce an extra pole. Recognizing this, we keep the poles closest to the plant's open loop poles, $-0.433 \pm 0.953j$. The feedback gains $\mathbf{K}$ can now be determined using MATLAB's place command,

$$\mathsf{K = place(F, G, [-0.433 + 0.953 * j; -0.433 - 0.953 * j]) = [2.09\ 0.87]}.$$

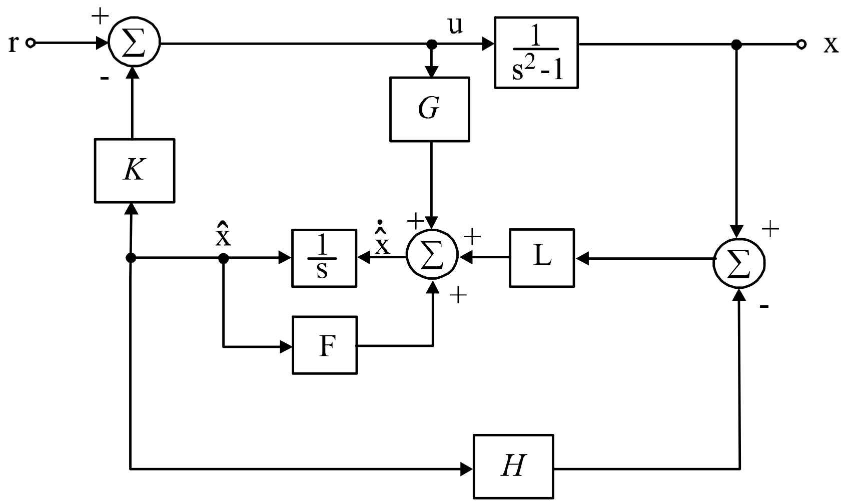
(f) The estimator gains are just as easy to produce. With $\zeta = 0.5$ and $\omega_n = 8$, we have,

$$\begin{aligned}
&\mathsf{[F, G, H, J] = tf2ss([0\ 0\ 1], [1\ 0\ \ -1])} \\
&\mathsf{pe = [1\ 2 * zeta * omegan\ omegan\hat{}2]} \\
&\mathsf{L} \quad = \quad \mathsf{place(F', H', pe)' = [8\ 65]}^T.
\end{aligned}$$

(g) The estimator equations are,

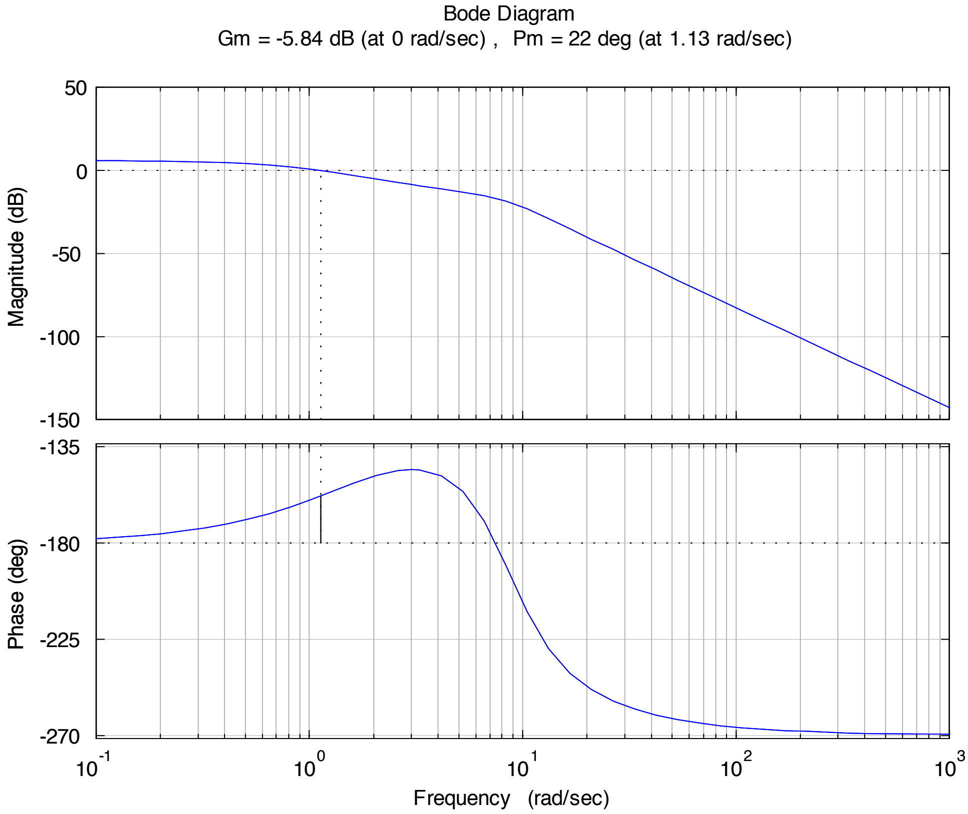
$$\begin{aligned}
\dot{\hat{\mathbf{x}}} &= \mathbf{F}\hat{\mathbf{x}} + \mathbf{G}u + \mathbf{L}(y - \mathbf{H}\hat{\mathbf{x}}), \\
u &= -\mathbf{K}\hat{\mathbf{x}}.
\end{aligned}$$

and are shown in block diagram form on top of the next page.



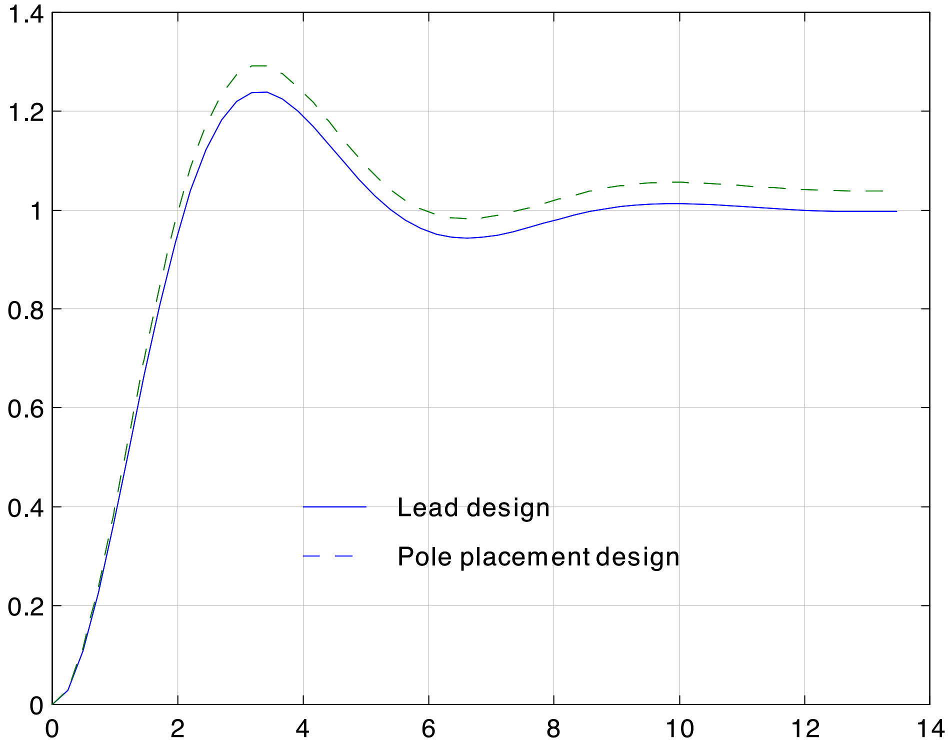
Block diagram of the combined estimator and control law in Problem 7.51.

The Bode plot of the controller and plant designed using pole placement techniques is shown below. The phase margin is approximately 22° and the gain margin now has a limitation both for increasing and decreasing the gain. The gain can be increased by a factor of $1/0.14 = 7.14 =$ 17 db and decreased by a factor of $1/1.96 = 0.51 = -5.8$ db. So the lead compensator has roughly equivalent stability margins.



Problem 7.51: Bode plot of plant and compensator design with pole placement.

The step responses for both designs are shown on the next page using the Matlab `step` command. They differ slightly because the DC gain of the compensator designed using pole placement hasn't been adjusted for unity gain. Also the specification for less than 25% overshoot has not been met with the pole placement design. This can be attributed to an estimator roots which are too slow. Increasing the ω_n of the estimator to 10 rad/sec will meet the specification.



Problem 7.51: Closed-loop step responses.

52. A simplified model for the control of a flexible robotic arm is shown in Fig. 7.98, where

$$k/M = 900 \text{ rad/sec}^2,$$
$$y = \text{output, the mass position},$$
$$u = \text{input, the position of the end of the spring}.$$

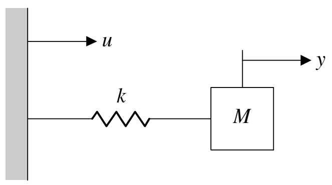


Figure 7.98: Simple robotic arm for Problem 7.52.

a) Write the equations of motion in state-space form.
b) Design an estimator with roots as $s = -100 \pm 100j$.